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THE GROUP OF HOMEOMORPHISMS OF A SOLENOID(1)

BY

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ABSTRACT. Let X be a topological space. An n-mean on X is a continuous function $\mu: X^n \to X$ which is symmetric and idempotent. In the first part of this paper it is shown that if X is a compact connected abelian topological group, then X admits an n-mean if and only if $H^1(X, Z)$ is n-divisible where $H^m(X, Z)$ is m-dimensional Čech cohomology with integers Z as coefficient group. This result is used to show that if Σ_a is a solenoid and $\mathrm{Aut}(\Sigma_a)$ is the group of topological group automorphisms of Σ_a, then $\mathrm{Aut}(\Sigma_a)$ is algebraically $Z_2 \times G$ where G is $\{0\}$, Z^n, or $\oplus_{i=1}^{\infty} Z$. For a given Σ_a, the structure of $\mathrm{Aut}(\Sigma_a)$ is determined by the n-means which Σ_a admits. Topologically, $\mathrm{Aut}(\Sigma_a)$ is a discrete space which has two points or is countably infinite.

The main result of the paper gives the precise topological structure of the group of homeomorphisms $G(\Sigma_a)$ of a solenoid Σ_a with the compact open topology. In the last section of the paper it is shown that $G(\Sigma_a)$ is homeomorphic to $\Sigma_a \times l_2 \times \mathrm{Aut}(\Sigma_a)$ where l_2 is separable infinite-dimensional Hilbert space. The proof of this result uses recent results in infinite-dimensional topology and some techniques using flows developed by the author in a previous paper.

Introduction. Let X be a topological space. An *n-mean* for $n \geq 2$ on X is a continuous function $\mu: X^n \to X$ having the property that $\mu(x_1, \cdots, x_n) = \mu(x_{\pi(1)}, \cdots, x_{\pi(n)})$ for any permutation π of $\{1, \cdots, n\}$ and $\mu(x, x, \cdots, x) = x$ for all x in X. We say simply that μ is symmetric and idempotent, respectively. Aumann showed that the circle T does not admit an n-mean for any n in [1]. In Eckmann [3] and Eckmann, Ganea, and Hilton [4] this result was extended to show that many other spaces do not support n-means. In particular, in [4] it is shown that if X is a compact connected polyhedron and X admits an n-mean for some n, then X is contractible. For the most part the above authors have devoted their efforts to widening the class of spaces known not to admit an n-mean. In the first section of this paper we show that a large class of compact connected abelian groups admit n-means for various n's. We give necessary and sufficient conditions that a compact connected abelian topological group H admit an n-mean. Among the equivalent conditions we show that H admits an n-mean if and only if

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$H^1(H, Z)$ is n-divisible where $H^m(H, Z)$ is the m-dimensional Čech cohomology with the integers Z as the coefficient group. The proof of the result makes use of theorems of Sigmon [13] and Steenrod [15].

The second section of the paper deals with the group of topological group automorphisms $\mathrm{Aut}(H)$ of a compact connected abelian topological group H. For a finite-dimensional H it is shown that $\mathrm{Aut}(H)$ is a subgroup of the multiplicative group of nonsingular $n \times n$ matrices with rational entries. In this case the compact open topology on $\mathrm{Aut}(H)$ is discrete. If H is a solenoid Σ_a, $\mathrm{Aut}(\Sigma_a)$ has the form Z_2 or $Z_2 \times \bigoplus_{i \in I} Z$ where I is finite or countably infinite. Actually, the cardinality of I is just the number of distinct prime numbers p for which Σ_a admits a p-mean.

In the last section of the paper the group of homeomorphisms $G(\Sigma_a)$ of a solenoid Σ_a is analyzed. It is shown that with the compact open topology $G(\Sigma_a)$ is homeomorphic to $\Sigma_a \times l_2 \times \mathrm{Aut}(\Sigma_a)$ where l_2 is the separable infinite-dimensional Hilbert space. The proof makes use of techniques using flows developed by the author in [8]. We also use a theorem of Scheffer [12] and several results in infinite-dimensional topology [2] and [6].

Notation. Let X be a topological space and let H be an abelian group. Then $H^n(X, H)$ is the n-dimensional Čech cohomology of X with H as coefficient group. The groups H that will be used as coefficients in this paper are the integers Z and $Z_n = \{0, 1, 2, \cdots, n-1\}$ with addition modulo n.

If G is a locally compact abelian topological group, then char G denotes the *character group* of G. The fundamental facts about the character group and Pontryagin duality are assumed to be known. Good references are the appropriate sections of [7] and [11]. If $g: G \to H$ is a group homomorphism between two locally compact abelian topological groups, then $g^*: \mathrm{char}\, H \to \mathrm{char}\, G$ denotes the dual map.

A word is in order about the various ways one can view solenoids. There are four equivalent points of view. If $a = (n_i)$ is a sequence of integers $n_i \geq 2$, then (1) the *a-adic solenoid* is $\Sigma_a = \varprojlim \{T_i, f_i\}$ where $T_i = T = \{z: z$ is a complex number with $|z| = 1\}$ and where $f_i: T_{i+1} \to T_i$ is defined by $f_i(z) = z^{n_i}$. Equivalently, (2) a definition of the a-adic solenoid is given in Hewitt and Ross [7, p. 114] which we will find useful. Using Pontryagin duality we could define a solenoid as (3) a compact connected abelian topological group G such that char G has rank one and is not finitely generated (see [11, Theorem 47, p. 259]). Since every torsion free abelian group of rank one is isomorphic to a subgroup of the rationals Q we could say (4) a solenoid is the character group of any subgroup of Q which is not cyclic. These viewpoints of solenoids will be used interchangeably in the paper.

An additive abelian group H is *n-divisible* provided that, for each $x \in H$, there is a $y \in H$ such that $ny = x$. If there is only one such y for each x, then H is uniquely n-divisible. A torsion free abelian group which is n-divisible is uniquely n-divisible.

The *rank* of an additive abelian group H is the maximum cardinality of a collection $A \subset H$ such that for every finite set $\{a_1, \cdots, a_n\} \subset A$ if $\Sigma_{i=1}^n n_i a_i = 0$ for a collection of integers $\{n_1, \cdots, n_n\}$ then $n_i = 0$ for all i. If $\{H_\alpha : \alpha \in A\}$ is a collection of additive abelian groups, then their *free sum* will be denoted by

$$\bigoplus_{\alpha \in A} H_\alpha = \left\{ (x_\alpha) \in \prod_{\alpha \in A} H_\alpha : x_\alpha = 0 \text{ except for a finite number of } \alpha\text{'s} \right\}.$$

If H is a torsion free abelian group with rank H = card A, then there is a group imbedding of H into $\bigoplus_{\alpha \in A} Q_\alpha$ where $Q_\alpha = Q$ for all α (see [7, Theorem A. 14, p. 444]).

Let X and Y be topological spaces and let $C(X, Y)$ be the set of continuous maps from X to Y with the compact open topology. If $A \subset X$ and $B \subset Y$ and $F(X, Y) \subset C(X, Y)$, then $(A, B) = \{f \in F(X, Y) : f[A] \subset B\}$. Thus (A, B) depends on the particular $F(X, Y)$ under discussion. If K is compact in X and V is open in Y, then the compact open topology on $F(X, Y)$ has as a subbasis sets of the form (K, V).

The symbol "$\sim$" means "homeomorphic to"; "$\simeq$", "isomorphic to".

1. **Means on topological groups.** Let X be a topological space and n a positive integer. An *n-mean* on X is a continuous function $\mu : X^n \to X$ having the property that $\mu(x_1, \cdots, x_n) = \mu(x_{\pi(1)}, \cdots, x_{\pi(n)})$ for any permutation π of $\{1, \cdots, n\}$ and $\mu(x, x, \cdots, x) = x$ for all $x \in X$. If G is a group, then a *homomorphic n-mean* is a function $\mu: G^n \to G$ which is symmetric and idempotent and which is in addition a group homomorphism. One can easily show that a group G admits a homomorphic n-mean if and only if G is abelian and uniquely n-divisible. If G is written additively, then any homomorphic m-mean must be of the form $\mu(x_1, \cdots, x_n) = n^{-1}(x_1 + \cdots + x_n)$ and thus is unique. It follows easily that if G is a compact topological group whose underlying group admits a homomorphic n-mean μ, then μ is also continuous and thus a topological n-mean. Our basic result in this section is the following theorem.

1.1 **Theorem.** *Let G be a compact connected abelian topological group. Then the following are equivalent*:

(1) *G admits a topological n-mean,*

(2) *G admits a homomorphic n-mean,*

(3) $H^1(G, Z_n) = 0$,

(4) $H^1(G, Z)$ *is n-divisible,*

(5) char G *is n-divisible,*

(6) $H^1(G, Z)$ *admits a homomorphic n-mean, and*

(7) char G *admits a homomorphic n-mean.*

The proof of this theorem is simply an observation based on following theorems. We will state the theorems and then proceed with the proof of Theorem 1.1.

1.2. **Theorem** (**Sigmon** [13]). *Let X be a continuum. Then if X admits an n-mean, then $H^1(X, Z_n) = 0$.*

1.3. **Theorem** (**Steenrod** [15]). *Let G be a compact connected abelian topological group. Then $H^1(G, Z) \simeq$ char G.*

The next result that we will use requires some justification. If H is any torsion free abelian group, then $H \otimes Z_n = \{0\}$ if and only if H is n-divisible. For a connected compact abelian group G, char G is torsion free, and $H^1(G, Z)$ is also by Theorem 1.3. By the universal coefficient theorem for Čech cohomology [14, p. 335] we have

1.4. **Proposition**. *Let G be a compact connected abelian topological group. Then $H^1(G, Z)$ is n-divisible if and only if $H^1(G, Z_n) = 0$.*

Proof of Theorem 1.1. By Theorem 1.3, (4) and (5) are equivalent and (6) and (7) are equivalent. By Proposition 1.4, (3) and (4) are equivalent. Since char G is torsion free and abelian, (5) and (7) are equivalent by the remarks made earlier about homomorphic n-means. Thus (3), $\cdots$, (7) are equivalent. Now (1) implies (3) by Theorem 1.2 and clearly (2) implies (1). It will thus be sufficient to show that (7) implies (2) and the theorem will be proved. Suppose that char G admits a homomorphic n-mean μ: $(\text{char } G)^n \longrightarrow \text{char } G$. Let Δ_n: char $G \longrightarrow (\text{char } G)^n$ be the diagonal map $\Delta_n(x) = (x, x, \cdots, x)$. We may consider the dual map μ^* as going from G to G^n by the duality theorem. Now $\mu^*[G]$ is contained in the diagonal of G^n since μ is symmetric. Also $\Delta_n^* \circ \mu^* = (\mu \circ \Delta_n)^*$ is the identity on G since $\mu \circ \Delta_n$ is the identity on char G. From the definition of the dual map $\Delta_n^*(x_1, \cdots, x_n) = x_1 + \cdots + x_n$. Thus $\mu^*(x) = (y, y, \cdots, y)$ for some $y \in G$ and $\Delta_n^*(y, y, \cdots, y) = y + y + \cdots + y = ny = \Delta_n^*(\mu^*(x)) = x$. That is, $ny = x$. Let $h(x) \equiv y$ where $\mu^*(x) = (y, y, \cdots, y)$. Then M: $G^n \longrightarrow G$ defined by $M(x_1, \cdots, x_n) = h(x_1) + \cdots + h(x_n)$ is clearly a homomorphic n-mean. Theorem 1.1 is now completely proved.

1.5. **Corollary**. *If G is a compact connected abelian topological group which admits an n-mean and H is a closed connected subgroup of G, then H admits an n-mean.*

Proof. Let e: $H \longrightarrow G$ be a topological group imbedding of H into G. Then e^*: char $G \longrightarrow$ char H is onto. Thus char H is n-divisible since char G is. Thus H admits an n-mean by Theorem 1.1.

1.6. **Corollary**. *Suppose that G is a locally compact abelian topological group which admits an n-mean for some n. Then all homotopy groups $\pi_k(G)$ are zero for $k = 1, 2, \cdots$.*

Proof. If G admits an n-mean, then so does the identity component G_0 of G. Now G_0 is compactly generated and thus, by the structure theorem for locally compact abelian compactly generated topological groups, $G_0 \simeq R^n \times L$ where L is a compact connected abelian topological group [11, Theorem 51, p. 269]. That is, G_0 has the homotopy type of a compact connected abelian topological group L. By Corollary 3 of [12], $\pi_1(L) \simeq \mathrm{Hom}(T, L)$. If $\mathrm{Hom}(T, L) \neq 0$, then there is an imbedding of T into L as a closed subgroup. This is impossible by Corollary 1.5 since T does not admit an n-mean. Thus $\pi_1(L) = \pi_1(G) = 0$. Also by Corollary 3 of [12], $\pi_k(G) = 0$ for all $k \geq 2$ and the theorem is proved.

1.7. **Example.** As a contrast to Corollary 1.6, there are connected abelian topological groups which admit continuous homomorphic n-means and have nonzero fundamental groups. In fact, let H be any n-divisible abelian group. Let H be given the discrete topology. In the notation of [9], if $G = H^*/H$, then using the techniques of [9] one can show that G is a completely metrizable connected locally contractible abelian topological group which admits a continuous homomorphic n-mean with $\pi_1(G) = H$. If H is countable, then G is a separable Fréchet manifold.

1.8. **Remark.** Theorem 1.1 is particularly useful in constructing spaces which admit n-means. For instance, it follows from the proof of Theorem 2.5 in the next section that if $\mathcal{P}_0$ is any subset of the set of all prime numbers $\mathcal{P}$ such that $\mathcal{P} - \mathcal{P}_0$ is infinite, then there are $2^{\aleph_0}$ solenoids no two of which are homeomorphic such that each one admits a p-mean for $p \in \mathcal{P}_0$ and does not admit a p-mean for $p \notin \mathcal{P}_0$.

2. **Automorphism groups.** Let G be a compact connected abelian topological group. Here we study $\mathrm{Aut}(G)$, the group of all topological group automorphisms of G with the compact open topology. Using Pontryagin duality one can easily see that $\mathrm{Aut}(G)$ is group isomorphic to the group of all group automorphisms of char G under the map which takes g to $(g^*)^{-1}$.

2.1. **Proposition.** *Let* card A *be the same as the rank of* char G. *Then* $\mathrm{Aut}(G)$ *is group isomorphic to a subgroup of the group of all group automorphisms of* $\bigoplus_{\alpha \in A} Q_\alpha$.

Proof. Let $\{x_\alpha\}_{\alpha \in A}$ be a maximal set of linearly independent elements of char G. Let F: char $G \to \bigoplus_{\alpha \in A} Q_\alpha$ where $F(x_\alpha) = y_\alpha = (\delta_{\alpha\beta}) \in \bigoplus_{\beta \in A} Q_\beta$ where $\delta_{\alpha\beta} = 0$ if $\alpha \neq \beta$ and $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$. Then F extends to an imbedding of char G into $\bigoplus_{\alpha \in A} Q_\alpha$. Identify char G with its image under the group imbedding F. One can show that any automorphism of char G can be extended to an automorphism of $\bigoplus_{\alpha \in A} Q_\alpha$. That extension will be unique since $\{x_\alpha\}_{\alpha \in A}$ is a maximal set of linearly independent elements in $\bigoplus_{\alpha \in A} Q_\alpha$. The map K taking each automorphism of char G to its unique extension to an automorphism of $\bigoplus_{\alpha \in A} Q_\alpha$ is a group imbedding. The proposition is now proved.

2.2. **Proposition.** *If* $\dim G = n < \infty$, *then* $\mathrm{Aut}(G)$ *is group isomorphic to a subgroup of the multiplicative group of nonsingular* $n \times n$ *matrices with rational entries.*

Proof. In Proposition 2.1, $\mathrm{Aut}(G)$ is isomorphic to a subgroup of $\mathrm{Gl}(n, Q)$ and thus has the form asserted.

2.3. **Corollary.** *If* $\dim G < \infty$, *then* $\mathrm{Aut}(G)$ *is finite or countably infinite and has the discrete topology.*

Proof. Clearly, $\mathrm{Aut}(G)$ is finite or countably infinite by Proposition 2.1. Now $\mathrm{Aut}(G)$ is a closed subgroup of $G(G)$, the group of homeomorphisms of G. Now G must be metrizable, since its character group is countable. It is well known that the group of homeomorphisms of a compact metric space has a complete metric. Thus $\mathrm{Aut}(G)$ has a complete metric also. However, a countable complete metric space which is homogeneous must be discrete. Thus $\mathrm{Aut}(G)$ must be discrete.

Turning now to solenoids, the automorphism group of a solenoid must be a subgroup of the group of all group automorphisms of Q by Proposition 2.1. The group of automorphisms of Q is just $Q - \{0\}$ under multiplication. This can be seen to be $Z_2 \times \bigoplus_{i=1}^{\infty} Z$. The next two theorems will show that the subgroups which are possible automorphism groups of solenoids are Z_2, $Z_2 \times Z^n$, and $Z_2 \times \bigoplus_{i=1}^{\infty} Z$. First we show the relationship between the automorphism group of a solenoid Σ_a and the n-means which Σ_a admits.

2.4. **Theorem.** *Let* n *be the number of prime numbers* p *for which the solenoid* Σ_a *admits a* p*-mean. Then* $\mathrm{Aut}(\Sigma_a)$ *is isomorphic to* (a) Z_2 *if* $n = 0$, (b) $Z_2 \times Z^n$ *if* n *is a positive integer, and* (c) $Z_2 \times \bigoplus_{i=1}^{\infty} Z$ *if* n *is infinite.*

Proof. It follows easily from the definition of an n-mean that if Σ_a admits an n-mean where n is a positive integer and $n = a \cdot b$, then Σ_a admits an a-mean and and a b-mean. Now let $A \subset Q$ be the character group of Σ_a. We may assume that $1 \in A$. Suppose that $g: A \to A$ is an automorphism of A. Then g extends to an automorphism g' of Q. But $g': Q \to Q$ is just multiplication by some nonzero rational number m/n where $n > 0$ and m and n are relatively prime. Let a and b be integers such that $ma + nb = 1$. Let $s \in A$. Then $as \in A$ and $bs \in A$. Also $(as)m/n \in A$ since $g'(as) = (as)m/n = g(as) \in A$. Thus $(as)m/n + bs = s/n \in A$. That is, A is n-divisible. By Theorem 1.1, Σ_a admits an n-mean and by the above remark Σ_a admits a p-mean for each prime divisor p of n.

Let $C \subset Q - \{0\}$ be the multiplicative group associated with $\mathrm{Aut}(\Sigma_a)$. What has been shown is that if $m/n \in C$ with $n > 0$ and m and n are relatively prime, then $1/p \in C$ for every prime divisor of n. By the same token, $1/p \in C$ for every prime divisor p of $|m|$ since $n/m = -n/-m \in C$. Therefore C is generated by the elements $\{\pm 1/p : p$ is prime and Σ_a admits a p-mean$\}$. Thus $\mathrm{Aut}(\Sigma_a)$ must have the form (a), (b), or (c) as asserted.

2.5. **Theorem.** *Let H be any of the groups* (a), (b), *or* (c) *in Theorem* 2.4. *Then there is a collection of* $2^{\aleph_0}$ *solenoids* $\{\Sigma_{a_\alpha}: \alpha \in A\}$ *no two of which are homeomorphic such that* $\mathrm{Aut}(\Sigma_{a_\alpha}) \simeq H$ *for all* $\alpha \in A$.

Proof. Let n be $0, 1, 2, \cdots$, or ∞. Let $\mathcal{P}_0$ be any collection of prime numbers such that card $\mathcal{P}_0 = n$ with $\mathcal{F} = \mathcal{P} - \mathcal{P}_0$ infinite where $\mathcal{P}$ is the set of all prime numbers. There is a collection of subsets $\{\mathcal{F}_\alpha: \alpha \in A\}$ of $\mathcal{F}$ having the property that (1) each $\mathcal{F}_\alpha$ is infinite, (2) $\mathcal{F}_\alpha \cap \mathcal{F}_\beta$ is finite for $\alpha \neq \beta$ in A, and (3) card $A = 2^{\aleph_0}$. For each $\alpha \in A$ let $B_\alpha \subset Q$ be the set defined by

$$B_\alpha = \{m/(p_1 \cdots p_n r_1 \cdots r_s): m \in Z,\ p_i \in \mathcal{F}_\alpha \text{ for } i = 1, \cdots, n$$

$$\text{with } p_i \neq p_j \text{ for } i \neq j, \text{ and } r_i \in \mathcal{P}_0 \text{ for } i = 1, \cdots, s\}.$$

Note that r_i may be equal to r_j for $i \neq j$ in our definition. Then B_α is a subgroup of Q. One can easily show that B_α is p-divisible for every $p \in \mathcal{P}_0$ and not p-divisible for every $p \in \mathcal{P} - \mathcal{P}_0$. Let $\Sigma_{a_\alpha} =$ char B_α where B_α has the discrete topology. Then $\mathrm{Aut}(\Sigma_{a_\alpha})$ is group isomorphic to H by Theorem 2.4. Suppose now that Σ_{a_α} and Σ_{a_β} are homeomorphic. Then $H^1(\Sigma_{a_\alpha}, Z)$ and $H^1(\Sigma_{a_\beta}, Z)$ are isomorphic as well. Thus $B_\alpha =$ char Σ_{a_α} and $B_\beta =$ char Σ_{a_β} are isomorphic by Theorem 1.3. We will now show that this implies that $\alpha = \beta$. Suppose that $\alpha \neq \beta$. Then let $g: B_\alpha \to B_\beta$ be an isomorphism. One can show that if $g(1) = m/n$, then $g(x) = mx/n$ for all $x \in B_\alpha$. Let $P = \{p_1, \cdots, p_s\}$ be the set of all prime divisors of $|m|$. Let $p \in \mathcal{F}_\alpha - \mathcal{F}_\beta \cup P$. Then $1/p \in B_\alpha$ and $g(1/p) = m/pn$. However, $m/pn \notin B_\beta$ which is a contradiction. Thus $\alpha = \beta$ and $\{\Sigma_{a_\alpha}: \alpha \in A\}$ is the required collection.

3. **The group of homeomorphisms of a solenoid.** Let us first state the main result of this section. If X is a topological space, then $G(X)$ is the group of homeomorphisms of X with the compact open topology.

3.1. **Theorem.** *Let Σ_a be a solenoid. Then $G(\Sigma_a)$ is homeomorphic to $\Sigma_a \times l_2 \times \mathrm{Aut}(\Sigma_a)$ where l_2 is a separable infinite-dimensional Hilbert space.*

Our theorem is only a topological one and it will be clear in the proof that the algebraic structure of $G(\Sigma_a)$ is much different than that of $\Sigma_a \times l_2 \times \mathrm{Aut}(\Sigma_a)$. However, the projection map $\pi: G(\Sigma_a) \to \mathrm{Aut}(\Sigma_a)$ is a topological group homomorphism so that, if A is the identity component of $G(\Sigma_a)$, then $\mathrm{Aut}(\Sigma_a) \simeq G(\Sigma_a)/A$.

Before we prove Theorem 3.1 it will be necessary to state some theorems to quote and prove some preliminary results. There are two fundamental theorems in infinite-dimensional topology which will be needed.. We first state these.

3.2. **Theorem (Anderson-Kadeč).** *Every separable infinite-dimensional Banach space is homeomorphic to R^∞.*

3.3. **Corollary.** *Let K be a nonempty compact metric space. Then $C(K, R^\infty)$ is homeomorphic to R^∞.*

Proof. Clearly $C(K, R^\infty) \sim C(K, R)^\infty \sim (R^\alpha)^\infty \sim R^\infty$ where $\alpha = n < \infty$ or $\alpha = \infty$.

Actually every separable infinite-dimensional Fréchet space is homeomorphic to R^∞, but Theorem 3.2 and its corollary will be sufficient for our purposes. A fairly elementary proof of Theorem 3.2 is given in [2].

A *separable Fréchet manifold* is a separable metric space X such that, for each $x \in X$, there is an open set U in X containing x such that $U \sim l_2$. A result of Henderson shows that if two separable Fréchet manifolds are homotopically equivalent, then they are homeomorphic [6]. In particular, the following is true:

3.4. **Theorem (Henderson).** *If F is a contractible separable Fréchet manifold, then F is homeomorphic to l_2.*

Stating the next theorem we need will require some definitions. Let G and H be topological groups. Let $\operatorname{Hom}(G, H)$ denote the set of all continuous group homomorphisms with the compact open topology. Let T denote the circle group which we will think of as the reals modulo the integers, R/Z. Let $p: R \to T$ be the natural map. If H is a locally compact abelian topological group, then $\operatorname{Hom}(H, T) \equiv \operatorname{char} H$, the character group of H. The duality theorem says that the map $P: H \to \operatorname{char}(\operatorname{char} H)$ defined by $P(x)(\chi) = \chi(x)$ for $x \in H$ and $\chi \in \operatorname{char} H$ is a topological group isomorphism. Let $q: \operatorname{Hom}(\operatorname{char} H, R) \to H$ be defined by $q(h) = p^{-1}(p \circ h)$. Let G be a compact connected topological group and let H be a locally compact abelian topological group. Define $C_e(G, H) = \{f: G \to H: f$ takes the identity e of G to $0 \in H\}$. Define $F: C_e(G, \operatorname{Hom}(\operatorname{char} H, R)) \times \operatorname{Hom}(G, H) \to C_e(G, H)$ by $F(f, h) = (q \circ f) + h$.

3.5. **Theorem** (Scheffer [12]). *Let G be a compact connected topological group and H a locally compact abelian topological group. Then F is an algebraic and topological isomorphism of $C_e(G, \operatorname{Hom}(\operatorname{char} H, R)) \times \operatorname{Hom}(G, H)$ onto $C_e(G, H)$.*

We will give one immediate application of Theorem 3.5 and then discuss certain aspects of the proof of Theorem 3.5 given in [12] in order to simplify the proof of Theorem 3.1.

3.6. **Theorem.** *Let G be a compact connected metrizable topological group and let H be a metrizable locally compact abelian topological group. Then $C(G, H)$ is homeomorphic to $H \times l_2 \times \operatorname{Hom}(G, H)$ and $\operatorname{Hom}(G, H)$ is 0-dimensional.*

Proof. Let $f \in C(G, H)$ and suppose that $f(e) = g \in H$ where e is the identity element of G. Let $L_g: H \to H$ be left translation by g. Then let $f' = L_{-g} \circ f$. Then $f' \in C_e(G, H)$. Define $P(f) = (g, f')$. Then $P: C(G, H) \to H \times C_e(G, H)$ is a homeomorphism. Now if $f \in C_e(G, H)$, then $f[G]$ is contained in the identity component H_0 of H. The group H_0 is closed in H and compactly generated. By a theorem due to Pontryagin $H_0 \simeq L \times R^n$ where L is a compact connected abelian topological group [11, Theorem 51, p. 269]. Thus $C_e(G, H) \sim C_e(G, H_0) \sim C_e(G, L \times R^n) \sim C_e(G, L) \times C_e(G, R^n)$. But $C_e(G, R^n)$ is a Banach space and homeomorphic to l_2. By Theorem 3.5, $C_e(G, L) \simeq C_e(G, \mathrm{Hom}(\mathrm{char}\, L, R)) \times \mathrm{Hom}(G, L)$. Now $\mathrm{Hom}(\mathrm{char}\, L, R)$ is isomorphic and homeomorphic to $C(A, R)$ where A is a discrete space with card A the same as the rank of char L. Since L is compact and metrizable, char L is countable and has rank $\leq \aleph_0$. If A is finite, then $C(A, R) \sim R^n$ for some n and if A is infinite, then $C(A, R) \sim R^\infty$. In either case, $C_e(G, H) \sim l_2 \times \mathrm{Hom}(G, H)$. Thus we have that $C(G, H) \sim H \times l_2 \times \mathrm{Hom}(G, H)$. We now show that $\mathrm{Hom}(G, H)$ is 0-dimensional. Clearly, $\mathrm{Hom}(G, H) = \mathrm{Hom}(G, L)$ since G is compact and connected. Since $\mathrm{Hom}(G, H)$ is a topological group, it is only necessary to show that there is a basis for the neighborhoods of $f_0 \equiv 0$ consisting of sets which are both open and closed. To show this let V be any open set containing 0 in L. Sets of the form $(G, V) = \{f \in \mathrm{Hom}(G, L): f[G] \subset V\}$ form a basis for the neighborhoods of f_0 in $\mathrm{Hom}(G, L)$. There is a continuous homomorphism $h: L \to T^n$ where T^n is the n-dimensional torus and $\ker h \subset V$. Let U be an open set in T^n containing 1 such that U does not contain any subgroup of T^n other than $\{1\}$. Let $O = h^{-1}(U)$. Then (G, O) has the property that $(G, O) = (G, \ker h)$ and is thus both open and closed in $\mathrm{Hom}(G, L)$ and is contained in (G, V). Thus $\mathrm{Hom}(G, L)$ is 0-dimensional at f_0 and thus is 0-dimensional.

3.7. **Remark.** Even though $\mathrm{Hom}(G, H)$ is 0-dimensional it may not be discrete. For example, if $G = T$ and $H = T^\infty$, then $\mathrm{Hom}(T, T^\infty) \simeq \mathrm{Hom}(T, T)^\infty \simeq Z^\infty$ is homeomorphic to the irrationals as a subspace of R.

We would like now to make a few remarks about Theorem 3.5 in order to make our application of the theorem easier. Let Σ_a be a solenoid. Then $\dim \Sigma_a = 1$ and Σ_a has a dense one-parameter subgroup $\phi: R \to \Sigma_a$. Now $\phi[R]$ is just the arc component of the identity element in Σ_a and the map ϕ is one-to-one, but not a topological imbedding. If $K \subset \phi[R]$ is a continuum, then K must be an arc. Consequently, using Theorem 3.2 and Theorem 7.2 of [5], one can show that $\phi^{-1}: K \to \phi^{-1}(K)$ is continuous. Actually R is just the associated locally arcwise connected group of $\phi[R]$ [5, Definition 3.3, p. 635].

Now consider $C_e(\Sigma_a, \Sigma_a)$ where $G = \Sigma_a = H$ in Theorem 3.5. Let $f \in C_e(\Sigma_a, \Sigma_a)$ and suppose that f is homotopic to $h \in \mathrm{Hom}(\Sigma_a, \Sigma_a)$. Let $g = f - h$. Then g is homotopic to $f_0 \equiv 0$ and thus $g[\Sigma_a] \subset \phi[R]$. Since $g[\Sigma_a]$ is compact and connected, by the remark made above, $\phi^{-1}: g[\Sigma_a] \to R$ is continuous. Define $B: C_e(\Sigma_a, \Sigma_a) \to \mathrm{Hom}(\Sigma_a, \Sigma_a) \times C_e(\Sigma_a, R)$ by $B(f) = (h, \phi^{-1} \circ g)$. Then B is equivalent to the map F^{-1} in Theorem 3.5 if one chooses an appropriate isomorphism between $\mathrm{Hom}(\mathrm{char}\, \Sigma_a, R)$ and R. Let $C_0(\Sigma_a)$ be the set of all $f \in C_e(\Sigma_a, \Sigma_a)$ such that f is homotopic to f_0. Then $C_0(\Sigma_a)$ is homeomorphic to $C_e(\Sigma_a, R)$ under the map $C(f) = \phi^{-1} \circ f$. Thus $C_e(\Sigma_a, \Sigma_a)$ and $\mathrm{Hom}(\Sigma_a, \Sigma_a) \times$

$C_0(\Sigma_a)$ are homeomorphic. What we basically need in the proof of Theorem 3.1 is the following lemma which we are now prepared to prove.

3.8. **Lemma.** *Let* $G_e(\Sigma_a) = \{f \in G(\Sigma_a): f(0) = 0\}$ *and let* $G_0(\Sigma_a) = \{f \in G_e(\Sigma_a)$: *$f$ is homotopic to the identity map*$\}$. *Then* $G_0(\Sigma_a)$ *is contractible.*

Proof. Let $C_0'(\Sigma_a) = \{f \in C_0(\Sigma_a): \mathrm{Id} + f \in G_e(\Sigma_a)\}$. Define $H: C_0(\Sigma_a) \times [0, 1] \to C_0(\Sigma_a)$ by $H(f, t) = \phi t \phi^{-1} \circ f \equiv f_t$. Now H is a contraction of $C_0(\Sigma_a)$ to f_0 by Theorem 3.5 together with the above remarks. We claim that $H|C_0'(\Sigma_a)$ is a contraction of $C_0'(\Sigma_a)$ in itself to f_0. When we have shown that, then the lemma will be proved by defining the contraction $P(h, t) = h_t$ where $h_t = \mathrm{Id} + f_t$ where $h = \mathrm{Id} + f$ with $f \in C_0'(\Sigma_a)$.

Claim 1. $H|C_0'(\Sigma_a)$ contracts $C_0'(\Sigma_a)$ in itself to f_0.

Proof of Claim 1. All that needs to be shown is that $H(f, t) \in C_0'(\Sigma_a)$ for all $t \in [0, 1]$ and all $f \in C_0'(\Sigma_a)$. Let $t \in (0, 1)$ and $f \in C_0'(\Sigma_a)$. Now $\mathrm{Id} + f = h$ is a homeomorphism by the definition of $C_0'(\Sigma_a)$. Suppose that $h_t = \mathrm{Id} + f_t$ is not a homeomorphism. Since $f: \Sigma_a \to \Sigma_a$ has the property that $f[\Sigma_a] \subset \phi[R]$ and $\phi^{-1} \circ f$ is continuous, it must be that $\phi^{-1} \circ f[\Sigma_a]$ is a bounded subset of R. Thus $t \cdot \phi^{-1} \circ f[\Sigma_a]$ is also a bounded subset of R. Therefore, $\phi^{-1} \circ h_t \circ \phi$: $R \to R$ is onto and $h_t: \Sigma_a \to \Sigma_a$ must also be onto. Since Σ_a is compact, if h_t were one-to-one, then h_t would be a homeomorphism which would be a contradiction. Thus h_t is not one-to-one and there is an $x \neq y$ in Σ_a with $h_t(x) = h_t(y)$.

Claim 2. We may suppose that $x = 0$ and that $y \in \phi[R]$.

Proof of Claim 2. Now $y - x \in \phi[R]$. Define $h'(z) = h(z + x) - h(x) = z + x + f(z + x) - h(x) = z + f(z + x) - f(x)$. Then $g(z) \equiv f(z + x) - f(x) \in C_0(\Sigma_a)$ and since h' is clearly a homeomorphism of Σ_a, $g \in C_0'(\Sigma_a)$. But $H(h', t)(y - x) = y - x + f_t(y) - f_t(x) = 0$. Thus $h_t'(0) = 0$ and $h_t'(y - x) = 0$ with $y - x \in \phi[R]$. Claim 2 now follows by renaming $f = g$, $h = h'$, $x = 0$, and $y = y - x$.

We now return to the proof of Claim 1. We suppose that $h_t(y) = 0$ with $y \neq 0$ and $y \in \phi[R]$. Consider the map $\hat{h}: R \to R$ defined by $\hat{h} = \phi^{-1} \circ h \circ \phi$. Then $\hat{h}$ is a homeomorphism of R onto itself using Theorem 3.2 of [5]. Let $\hat{f}: R \to R$ be defined by $\phi^{-1} \circ f \circ \phi$. Then $\hat{h} = \mathrm{Id} + \hat{f}$. Now $\hat{f}$ is bounded. Thus $\hat{h}$ must be an orientation preserving homeomorphism of R. (For if $\hat{h}(z) < \hat{h}(w)$ for all $z > w$, then $z - w < \hat{f}(w) - \hat{f}(z)$ for all $z > w$, and letting $z \to \infty$, $\hat{f}(z)$ could not be bounded below, a contradiction.) But then $\hat{h}_\tau = \mathrm{Id} + \tau \cdot \hat{f}$ is also a homeomorphism of R for all $\tau \in [0, 1]$. This contradicts the fact that $\hat{h}_t(\phi^{-1}(y)) = \hat{h}_t(0) = 0$ with $\phi^{-1}(y) \neq 0$. This contradiction establishes Claim 1. The proof of Lemma 3.8 is now complete.

Proof of Theorem 3.1. Let $G_0(\Sigma_a)$ be the set of all homeomorphisms of Σ_a onto itself which take 0 to 0 and such that, if $f \in G_0(\Sigma_a)$, then f is homotopic to the identity homeomorphism of Σ_a. We have shown in Lemma 3.8 that $G_0(\Sigma_a)$ is contractible. Since $G_0(\Sigma_a)$ is also a topological group, $G_0(\Sigma_a)$ is also locally contractible and thus locally connected and locally arcwise connected. The principal difficulty in the proof of Theorem 3.1 is showing that $G_0(\Sigma_a)$ is homeomorphic to l_2. However, we first show the following.

Claim 1. $G(\Sigma_a) \sim \Sigma_a \times G_0(\Sigma_a) \times \mathrm{Aut}(\Sigma_a)$.

Proof of Claim 1. Let $f \in G(\Sigma_a)$. Define the map $P: G(\Sigma_a) \to \Sigma_a \times G_e(\Sigma_a)$ by $P(f) = (f(0), L_{-f(0)} \circ f)$. Then P is a homeomorphism. We will now show that $G_e(\Sigma_a) \sim G_0(\Sigma_a) \times \mathrm{Aut}(\Sigma_a)$. Define $B: G_0(\Sigma_a) \times \mathrm{Aut}(\Sigma_a) \to G_e(\Sigma_a)$ by $B(f, g) = g \circ f$. By Theorem 3.5 and Lemma 3.8, B is onto. Because Σ_a is compact, B is continuous. Now $\mathrm{Aut}(\Sigma_a)$ is discrete by Corollary 2.3 and $B^{-1}|G_0(\Sigma_a)$ is the identity map. Thus B is a homeomorphism.

We now set about to show that $G_0(\Sigma_a)$ is homeomorphic to l_2. This will be done by a sequence of claims.

Claim 2. $G_0(\Sigma_a)$ is a separable Fréchet manifold.

Proof of Claim 2. It will be helpful to use the model of the a-adic solenoid given in §10 of [7]. Let $a = (n_i)$ be a sequence of integers $n_i \geq 2$. Then Δ_a is a topological group which is homeomorphic to the Cantor set [7, Definition 10.2, p. 109]. There is an element $u \in \Delta_a$ such that $\{nu: n \in Z\}$ is dense in Δ_a with $\Sigma_a = (R \times \Delta_a)/B$ where $B = \{(n, nu): n \in Z\}$ [7, Definition 10.12, p. 114]. Let ν: $R \times \Delta_a \to \Sigma_a$ be the natural map. If $a < b$ in R with $b - a = 1$, then $\nu|[a, b) \times \Delta_a$ and $\nu|(a, b) \times \Delta_a$ is one-to-one and onto Σ_a and $\nu|(a, b) \times \Delta_a$ is a homeomorphism onto its image in Σ_a. Since B is discrete, ν is a local homeomorphism and $\nu[(a,b) \times \Delta_a]$ is open in Σ_a. Let $\tilde{\Delta}_a = \nu[\{1/2\} \times \Delta_a]$ and $U = \nu[(0, 1) \times \Delta_a]$. Then $\tilde{\Delta}_a$ is compact and U is open in Σ_a. Let $\mathfrak{U} = (\tilde{\Delta}_a, U)$ and let $\mathcal{O}$ be the component of the identity map Id in $\mathfrak{U}$. Then $\mathcal{O}$ is open in $G_0(\Sigma_a)$ since $G_0(\Sigma_a)$ is locally connected as already remarked. We will now show that $\mathcal{O}$ is homeomorphic to l_2. Then Claim 2 will be proved since $G_0(\Sigma_a)$ is a topological group.

Claim 3. $\mathcal{O} \sim l_2$.

Proof of Claim 3. Recall that a *flow* is a topological group action of the additive reals on a topological space. To prove Claim 3 and later claims we will use the techniques using flows developed by the author in [8]. Some familiarity with [8] is assumed.

Now $\mathcal{O}$ is arcwise connected. Thus if $f \in \mathcal{O}$, then for each $x \in \Delta_a$, $f(\nu(1/2, x)) \in \nu[(0, 1) \times \{x\}]$. Let $x \in \Delta_a$ and define $f_x: [0, 1] \to \Sigma_a$ by $f_x(t) = \nu(t, x)$. Define a flow $F: R \times [0, 1] \to [0, 1]$ by $F(r, t) = t^{\exp(r)}$. Let $g \in \mathcal{O}$ and let $P(g)$: $\Sigma_a \to \Sigma_a$ be defined by $P(g)(f_x(t)) = f_x \circ F(r, t)$ where $r \in R$ is given by the relation $g(f_x(1/2)) = f_x \circ F(r, 1/2)$. Note that $P(g)$ is well defined since $g(f_x(1/2)) \in f_x[F[R \times \{1/2\}]]$ and $P(g)(f_x(t)) = f_x(t)$ for $t = 0$ or $t = 1$. Clearly, $P(g) \in G_0(\Sigma_a)$. Define $G_0'(\Sigma_a) = \{f \in G_0(\Sigma_a): f|\tilde{\Delta}_a \equiv \mathrm{Id}\}$. Let $\mathcal{F} = \{g \in G_0(\Sigma_a): g = P(g)\}$. Then $\mathcal{F} = \{g \in G_0(\Sigma_a)$: for each $x \in \Delta_a$, $f_x \circ F(r_x, t) = g(f_x(t))$ for all $t \in [0, 1]$ for some $r_x \in R\}$. Define $L: \mathcal{O} \to \mathcal{F} \times G_0'(\Sigma_a)$ by $L(g) = (P(g), P(g)^{-1} \circ g)$. One can verify that L is a homeomorphism which is onto. Now we will show that $\mathcal{F} \sim l_2$ and $G_0'(\Sigma_a) \sim l_2$. Then $\mathcal{O} \sim l_2$ and Claim 3 will be proved.

Claim 4. $\mathcal{F} \sim l_2$.

Proof of Claim 4. Let $g \in \mathcal{F}$ and define $K(g)\colon \Delta_a \to R$ by $K(g)(x) = r$ where $g(f_x(\frac{1}{2})) = f_x \circ F(r, \frac{1}{2})$. Then $K\colon \mathcal{F} \to C_0(\Delta_a, R) = \{f \in C(\Delta_a, R)\colon f(0) = 0\}$ is a homeomorphism. Clearly $C_0(\Delta_a, R)$ is a separable infinite-dimensional Banach space and thus is homeomorphic to l_2 by Theorem 3.2. Claim 4 is now proved.

Claim 5. $G_0'(\Sigma_a) \sim l_2$.

Proof of Claim 5. Let $I = [0, 1]$ be the closed unit interval and $H_0(I) = \{f \in G(I)\colon f$ is orientation preserving$\}$. Then $H_0(I) \sim l_2$ by a theorem which is due to Anderson. A proof of this is given in [8]. Let $x \in \Delta_a$ and define g_x: $[0, 1] \to \Sigma_a$ by $g_x(t) = \nu(t + \frac{1}{2}, x)$. If $\underset{\sim}{g} \in G_0'(\Sigma_a)$, then $g(g_x(0)) = g_x(0)$ and $g(g_x(1)) = g_x(1)$ for all $x \in \Delta_a$ since $g|\Delta_a \equiv \mathrm{Id}$. Thus for each $x \in \Delta_a$, $g_x^{-1} \circ g \circ g_x\colon I \to I$ is a homeomorphism of I which is isotopic to the identity and thus orientation preserving. Define $K(g)\colon \Delta_a \to H_0(I)$ by $K(g)(x) = g_x^{-1} \circ g \circ g_x$. Then $K(g) \in C(\Delta_a, H_0(I))$ and $K\colon G_0'(\Sigma_a) \to C(\Delta_a, H_0(I))$ is a homeomorphism. But $C(\Delta_a, H_0(I))$ can be made into a separable infinite-dimensional Banach space since $H_0(I) \sim l_2$. Thus $C(\Delta_a, H_0(I)) \sim l_2$ by Theorem 3.2. Thus $G_0'(\Sigma_a) \sim l_2$ and Claim 5 is proved.

Claim 3 now follows from Claim 4 and Claim 5, and is completely proved. Claim 3 completes the proof of Claim 2.

Since $G_0(\Sigma_a)$ is a contractible separable Fréchet manifold it must be homeomorphic to l_2 by Theorem 3.4. Thus, using Claim 1 it follows that $G(\Sigma_a)$ is homeomorphic to $\Sigma_a \times l_2 \times \mathrm{Aut}(\Sigma_a)$ and Theorem 3.1 is now proved.

3.9. **Corollary.** *Let Σ_a and Σ_b be solenoids. Then if $G(\Sigma_a)$ and $G(\Sigma_b)$ are homeomorphic, then Σ_a and Σ_b are isomorphic as topological groups.*

Proof. Since $G(\Sigma_a)$ and $G(\Sigma_b)$ are homeomorphic, $H^1(G(\Sigma_a), Z)$ and $H^1(G(\Sigma_b), Z)$ are isomorphic. However, from Theorem 3.1, $H^1(G(\Sigma_a), Z) \simeq H^1(\Sigma_a, Z)$ and $H^1(G(\Sigma_b), Z) \simeq H^1(\Sigma_b, Z)$. Thus $H^1(\Sigma_a, Z)$ and $H^1(\Sigma_b, Z)$ are isomorphic. By Theorem 1.3, char Σ_a and char Σ_b are isomorphic. By the duality theorem, Σ_a and Σ_b must be isomorphic as topological groups.

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